Math Review

Summer 2016

*Topic 1*

1. Introduction to mathematical notations and logic

Goals:

* Able to read proofs and definition comfortably
* Understand what a definition means and does not mean

Now you should be able to understand this:

Sample example from Consumer Theory:

is a family of nonempty subsets of . Every element of is a set is choice rule that assigns a nonempty set of chosen elements in B, i.e. .

Definition: The choice structure satisfies the *weak axiom of revealed preference*:

If for some with , we have , then for any with and , we must also have

We will come back to this in a bit after going through some notations.

**Demonstration:**

is OK

**OR**is ok too.

**How about** No, not ok.

Which of these violate the axiom listed here?

1. The choice structure is where the choice rule is and *(does not violate)*

2. 1. The choice structure is where the choice rule is and *(violates)*

3. 1. The choice structure is where the choice rule

, *(does not violate)*

* 1. Mathematical notation

We start by getting acquainted (or re-acquainted for some of you) to the basic mathematical notations that you will see in economics.

|  |  |
| --- | --- |
|  | For all... |
|  | There exists... |
|  | There does not exist... |
|  | Therefore... |
|  | Because... |
|  | Negation |
|  | Identical to or the same as… For example, we write if for all |
|  | means: “implies , “Ifthen or is sufficient condition for ” |
| ⇔ | A ⇔B means “A if and only if B”,“A is equivalent to B” or “A is a necessary and sufficient condition for B” |
|  | "B strictly contains A" or “A is a proper subset of B” |
|  | “B contains A” or “A is a subset of B” |
|  | In... (Not in...) or an element of... (Not an element of...) |
|  | Bonus: End of proof, Q.E.D. |

What QED means? Latin phrase quod erat demonstrandum, meaning "which is what had to be proved".

The last three notations deal with sets. Formally, a set is a collection of well-defined and distinct objects (usually numbers). For example, the set A is completely determined by the elements in A, where:

We will touch more on sets in the next section.

We will deal with sets quite a bit today, you can think of it as a collection of things, you call members. Well-defined means that you know what is a member and most importantly, who is NOT. Set theory says that it does not matter how the members of the set are arranged. Therefore, {2, 3, 4, 6, 8, 10} is identical to {10, 2, 4, 8, 6, 3}. It does not affect the well-defined nature of a set.

* 1. Numbers

The different sets of numbers in mathematics are:

*Natural numbers*:

*Integers:*

*Rational numbers*:

: What do you think is missing in this definition of rational numbers? It has something to do with q.

:

*Real numbers*:

*Complex numbers*:

* + 1. Intervals in

These are the four sets of intervals in the real line:

*Closed interval*:

*Open interval*:

*Right-half closed or left-half open*

*Other*:

where ∞ denotes infinity. We also have −∞ for negative infinity.

* 1. Necessity and sufficiency

Before, jumping into proofs, we establish what we really mean by necessity and sufficiency. Necessary and sufficient have two very different meanings.

- If you advance that “**A is necessary for B**,” this is what is entailed:

* "A is implied by B" (
* For B to be true requires A to be true or equally, A is required to have B. Every time you have B you will have A, without exception

*Example*. Let A be the set "x is an integer less than 9" and let B be the set "x is an integer less than 7". Then A is implied by B, because "x is an integer less than 9" is implied by the statement "x is an integer less than 7".

- If you advance that “**A is sufficient for B**,” this is what is entailed:

* "A implies B"
* Whenever A holds, B must hold. Every time you have A you will have B

*Example*. If Sally gets an 100% in all her graded assignments (**A**), she gets a pass in the class (**B**). Getting 100% in all assignments is a sufficient condition to pass the class. But Sally may very well get an 88% in Homework#7 and still get an A in the class.

Contrapositive form:

Suppose we know that is true. Then, as A is necessary for B, when A is not true, then B cannot be true.

: Look back at your table of notations, how can you write this contrapositive form for A and B?

:

*Example*. Let A="x is an integer less than 9" and B= "x is an integer less than 7". As we saw earlier, A is implied by B. Now, we form the contrapositives, so A= "x is not an integer less than 9" and B= "x is not an integer less than 7". This implies A ⇒B is a true statement.

* 1. Theorems and proofs

A mathematical proof is used to show the validity of a specified statement. A proof uses logic and deductive reasoning to show that the statement is always true. Proofs are usually statements take the form “if A then B.” There are three types of proofs that are frequently used. I have them down here by their popularity (in my opinion) in the first year micro series.

* + 1. Proof by contradiction

This is a very powerful form of proof. In a proof by contradiction you show that “if not B then not A.” Logically, this is what it means:

All these three statements are *all equivalent*.

A good proof by contradiction has the following steps:

*Step 1*: Assume B is false

*Step 2*: Show that A must also be false.

We start with a simple math example, and later we will go through a slightly more involved example from micro theory after completing Topic 2.

*Example*. Prove that is irrational.

We could jump to Steps 1 and 2 but let’s be a little more careful.

Define related concepts: Rational numbers are all numbers of the form , where and are integers and .

Think of some different examples: Rational numbers take the form of

Re-define the concept of interest: A number which is not rational is said to be irrational.

Proof:

*Step 1*: Assume to the contrary that is rational. Thus we can write

Moreover, let and have no common divisor > 1, that is, is in the lowest terms. Then we have:

which implies:

*Step 2*: Because we have . An integer is even if so, by definition, must be an even integer. The square of an odd number is always odd (square of even is even) so because is even it must also hold that is even. Because is even there exists such that

Thus:

You can apply the same logic as above to show that must also be even. But if both and are even then the fraction cannot be in lowest terms since both integers are divisible by 2, a contradiction. Therefore is has to be irrational.

* + 1. Proof by construction

In proof by construction you use true statements to construct the actual statement that you wish to prove. Suppose we have the theorem "A ⇒ B". Here, A is called the premise and B the conclusion. In a constructive proof we assume that A is true, deduce various consequences of that, and use them to show that B must also hold. This proof technique is a little less structured, as it is more dependent on the nature of the statement you are trying to prove.

Proof by construction follows these two steps:

*Step 1*: State what you wish to show (i.e. your claim)

*Step 2*: Use valid logic and parameters to construct the statement.

*Step 3*: Conclusion. This is optional, you can re-state the goal if desired.

*Example*: Prove that if and are consecutive integers, then the sum is odd.

Proof.

*Step 1*: Assume that and are consecutive integers.

*Step 2*: Because and are consecutive we know that . Thus, the sum may be re-written as . Thus, there exists a number such that ,

*Step 3*: So the sum is odd.

: How would you approach this simple proof as a proof by contradiction?

: Assume that and are consecutive integers.

Assume also that the sum is ***not*** odd.

Because the sum is not odd, there exists no number such that :

However, the integers and are consecutive, so we may write the sum as . Thus, we have derived that for any integer and also that . This is a contradiction. If we hold that and are consecutive integers then we know that the sum must be odd.

* + 1. Proof by Induction

Proof by induction is another great method in which we use recursion to demonstrate an infinite number of facts in a finite amount of space. In other words, you wish to show that some statement, , is true for all , . To prove this general statement with induction we follow two steps:

*Step 1*: Show that a propositional form is true for some basis case. It is typical to begin by showing that either is true or is true for example.

*Step 2*: Assume that is true for some. This assumption is called the inductive hypothesis. Prove that is also true, using the assumption that is true.

I will skip this proof by induction and leave it as a reading material for you *(if we have time we will reference this again when going through tips for writing proofs)*

*Example*. Prove that for all .

Proof:

*Step 1*: Consider the case where . Then we have:

and the statement holds.

*Step 2.a*: Now assume that for some We have the following hold:

*Step 2.b*: We now wish to show that is true for , that is:

=?

We know from our inductive hypothesis that . Plugging this into the equation into the above, we have:

Thus we have shown that the statement also holds for which implies that for all .

General tips for approaching and writing proofs:

* As much as possible, use complete sentences when writing your proof. When writing a proof for a homework, exam or prelim, be as legible as possible. This holds for all parts of submitted work, and especially for proofs.
* Always remember to define any variables you introduce.
* It’s a good practice to say what type of proof you are using (e.g. Proof by contradiction) to help your reader.
* Overly wordy proofs may result in more likelihood for errors – keep things concise and simple.
* Avoid the use of words such as *obviously*, *clearly*, *as we know*, etc. State what is clear and obvious to you as it may not be for the reader. You might see these words in your micro notes, but I would personally stay clear of these.
* If asked to prove , that is “A if and only if B” then you must remember to complete both directions of the proof. You must prove “*if A then B*” and “*if B then A.*”

1. Prove: The sum of two even integers is always even. (*hint: use definition of even numbers*)

Proof: Let and be any two even integers, so there exist integers and such that and . Then, ), which is even.

2. Prove: Suppose and are integers and . If does not divide , then the equation has no positive integer solution. (*hint: use quadratic formula*)

Proof. Suppose is an integer such that .

Then, the quadratic formula tells us

Because , we must have

Thus, and divides .